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Institute of Communications Engineering

Comparing Approaches to the Spherical and Planar Single Layer Potentials for Interior Sound Field Synthesis

Frank Schultz and Sascha Spors

University of Rostock
Institute of Communications Engineering

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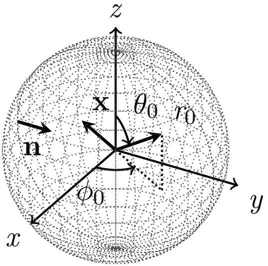
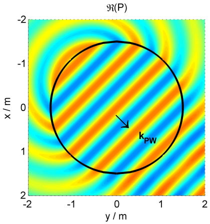
Introduction

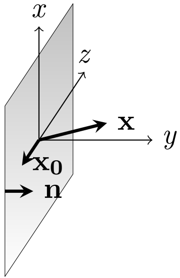
Sound field synthesis with **weighted** spherical **monopoles** located on a **continuous** secondary source distribution (SSD) $\rightarrow$ single layer potential

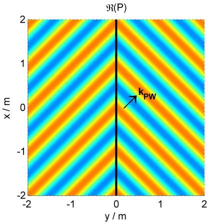
$$P(\mathbf{x},\omega) = \oint_{\partial V} D(\mathbf{x_0},\omega) \underbrace{\frac{1}{4\,\pi}\,\frac{\mathrm{e}^{-\mathrm{j}\,\frac{\omega}{c}\,|\mathbf{x}-\mathbf{x_0}|}}{|\mathbf{x}-\mathbf{x_0}|}}_{\text{symmetric kernel } G(|\mathbf{x}-\mathbf{x_0}|,\omega)} \,\mathrm{d}A(\mathbf{x_0})$$

How to find the **unknown** weights $D(\mathbf{x_0},\omega)$ for a **desired** (known) sound field $P(\mathbf{x},\omega)$ with **different analytic** methods? $\rightarrow$ solution of an inverse problem









Approaches to the Single Layer Potential

1. **Mathematical motivation → Fredholm integral of first kind**

$$P(\mathbf{x}) = \oint_{\partial V} D(\mathbf{x_0})\, G(|\mathbf{x} - \mathbf{x_0}|)\, \mathrm{d}A(\mathbf{x_0})$$

2. Physical motivation → interpretation of Helmholtz integral equation

$$\alpha\, P(\mathbf{x}) = \oint_{\partial V} \left[-\frac{\partial S(\mathbf{x_0})}{\partial n}\, G(|\mathbf{x} - \mathbf{x_0}|) + \frac{\partial G(|\mathbf{x} - \mathbf{x_0}|)}{\partial n}\, S(\mathbf{x_0}) \right] \mathrm{d}A(\mathbf{x_0})$$

with unit inward $\mathbf{n}$ and

$$\alpha = \begin{cases} +1 & \forall \mathbf{x} \in V \\ +1/2 & \forall \mathbf{x} \in \partial V \\ 0 & \forall \mathbf{x} \notin V \end{cases}$$

Approaches to the Single Layer Potential

1. **Mathematical motivation**

$$P(\mathbf{x}) = \oint_{\partial V} D(\mathbf{x_0})\, G(|\mathbf{x} - \mathbf{x_0}|)\, \mathrm{d}A(\mathbf{x_0})$$

- the single layer potential is a **Fredholm integral** of first kind
- **explicit** solution for the unknown driving function $D(\mathbf{x_0})$

spherical problem $\rightarrow$ **closed** SSD $\rightarrow$ compact Fredholm operator [Spo08b, Faz10]

$$P(\mathbf{x}) = \int_0^{2\pi} \int_0^{\pi} D(\mathbf{x_0})\, G(|\mathbf{x} - \mathbf{x_0}|)\, r_0^2\, \sin\theta_0\, \mathrm{d}\theta_0 \mathrm{d}\phi_0$$

planar problem $\rightarrow$ **infinite** SSD $\rightarrow$ Fourier operator [Ahr10, Faz10]

$$P(\mathbf{x}) = \iint_{-\infty}^{+\infty} D(\mathbf{x_0})\, G(|\mathbf{x} - \mathbf{x_0}|)\, \mathrm{d}x_0\, \mathrm{d}z_0$$

Solving the Fredholm Integral for Spherical SSD

Ambisonics [Ahr08, Faz12]

$$P(\mathbf{x}) = \int_0^{2\pi} \int_0^{\pi} D(\mathbf{x_0})\; G(|\mathbf{x} - \mathbf{x_0}|)\; r_0^2\; \sin\theta_0\, \mathrm{d}\theta_0 \mathrm{d}\phi_0$$

- expansion with ortho-normalized set that fulfills the wave equation, i.e. $Y_n^m(\theta, \phi)$
- **infinite** and **denumerable** eigenvalues

$$G(|\mathbf{x} - \mathbf{x_0}|) = \sum_{n=0}^{\infty} \sum_{m=-n}^{+n} -\mathrm{j}\, \frac{\omega}{c}\, h_n^{(2)}(\frac{\omega}{c}\, r_0)\, j_n(\frac{\omega}{c}\, r)\; Y_n^m(\theta, \phi)\; Y_n^m(\theta_0, \phi_0)^*$$

$$\begin{Bmatrix} S(\mathbf{x}) \\ D(\mathbf{x}) \end{Bmatrix} = \sum_{n=0}^{\infty} \sum_{m=-n}^{+n} \begin{Bmatrix} S_n^m(r) \\ D_n^m(r) \end{Bmatrix}\; Y_n^m(\theta, \phi)$$

$$D(\mathbf{x_0}) = \sum_{n=0}^{\infty} \sum_{m=-n}^{+n} \frac{1}{r_0^2} \underbrace{\frac{S_n^m(r)}{-\mathrm{j}\, \frac{\omega}{c}\, h_n^{(2)}(\frac{\omega}{c}\, r_0)\, j_n(\frac{\omega}{c}\, r)}}_{D_n^m(r)}\; Y_n^m(\theta_0, \phi_0)$$

Solving the Fredholm Integral for Planar SSD

Spectral Division Method [Ahr10, Faz10]

$$P(\mathbf{x}) = \iint\limits_{-\infty}^{+\infty} D(\mathbf{x_0})\; G(|\mathbf{x}-\mathbf{x_0}|)\,\mathrm{d}x_0\,\mathrm{d}z_0$$

- expansion with orthogonal, δ-normalized set that fulfills the wave equation, i.e. $\mathrm{e}^{\pm\mathrm{j}\,\langle\mathbf{k},\mathbf{x}\rangle}$
- **infinite** and **nondenumerable** 'eigenvalues' $\rightarrow$ **continuous** angular spectrum

$$G(|\mathbf{x}-\mathbf{x_0}|) = \frac{1}{(2\,\pi)^2}\iint\limits_{-\infty}^{+\infty} -\mathrm{j}\,\frac{\mathrm{e}^{-\mathrm{j}\,k_y\cdot y}}{2\,k_y}\,\mathrm{e}^{-\mathrm{j}(k_x\,x+k_z\,z)}\,\mathrm{e}^{+\mathrm{j}(k_x\,x_0+k_z\,z_0)}\mathrm{d}k_x\,\mathrm{d}k_z$$

$$\begin{Bmatrix} S(\mathbf{x}) \\ D(\mathbf{x}) \end{Bmatrix} = \frac{1}{(2\,\pi)^2}\iint\limits_{-\infty}^{+\infty} \begin{Bmatrix} S(k_x,y,k_z) \\ D(k_x,y,k_z) \end{Bmatrix}\,\mathrm{e}^{-\mathrm{j}\,(k_x\,x+k_z\,z)}\mathrm{d}k_x\,\mathrm{d}k_z$$

$$D(\mathbf{x_0}) = \frac{1}{(2\,\pi)^2}\iint\limits_{-\infty}^{+\infty} \frac{S(k_x,y,k_z)}{-\mathrm{j}\,\frac{\mathrm{e}^{-\mathrm{j}\,k_y\cdot y}}{2\,k_y}}\,\mathrm{e}^{-\mathrm{j}\,(k_x\,x_0+k_z\,z_0)}\mathrm{d}k_x\,\mathrm{d}k_z$$

Approaches to the Single Layer Potential

1. Mathematical motivation $\rightarrow$ Fredholm integral of first kind

$$P(\mathbf{x}) = \oint_{\partial V} D(\mathbf{x_0})\, G(|\mathbf{x} - \mathbf{x_0}|)\, \mathrm{d}A(\mathbf{x_0})$$

2. **Physical motivation $\rightarrow$ interpretation of Helmholtz integral equation**

$$\alpha\, P(\mathbf{x}) = \oint_{\partial V} \left[\underbrace{-\frac{\partial S(\mathbf{x_0})}{\partial n}}_{\text{normal velocity}} \underbrace{G(|\mathbf{x} - \mathbf{x_0}|)}_{\text{monopole}} + \underbrace{\frac{\partial G(|\mathbf{x} - \mathbf{x_0}|)}{\partial n}}_{\text{dipole}} \underbrace{S(\mathbf{x_0})}_{\text{pressure}} \right] \mathrm{d}A(\mathbf{x_0})$$

with unit inward $\mathbf{n}$ and

$$\alpha = \begin{cases} +1 & \forall \mathbf{x} \in V \\ +1/2 & \forall \mathbf{x} \in \partial V \\ 0 & \forall \mathbf{x} \notin V \end{cases}$$

Boundary Conditions for the Helmholtz Integral Equation

$$\alpha\, P(\mathbf{x}) = \oint_{\partial V} \left[-\frac{\partial S_T(\mathbf{x_0})}{\partial n}\, G(|\mathbf{x}-\mathbf{x_0}|) + \frac{\partial G(|\mathbf{x}-\mathbf{x_0}|)}{\partial n}\, S_T(\mathbf{x_0}) \right] \mathrm{d}A(\mathbf{x_0})$$

- desired sound field $S(\mathbf{x}) = P_{\text{IN}}(\mathbf{x})$ for $\mathbf{x} \in V$
- 'total' sound field $S_T(\mathbf{x}) = P_{\text{IN}}(\mathbf{x}) + P_{\text{SC}}(\mathbf{x})$ for $\mathbf{x} \notin V$

	Dirichlet	Neumann
$S_T(\mathbf{x_0})$	$S_T(\mathbf{x_0}) = 0$ equivalent sound-soft scattering [Faz13, Zot13]	$\frac{\partial S_T(\mathbf{x_0})}{\partial n} = 0$ equivalent sound-hard scattering [Faz12]
$\|G(\mathbf{x}-\mathbf{x_0}\|)$	$G_D(\mathbf{x}, \mathbf{x_0}) = 0$ Rayleigh II for planar SSD initial WFS research @ TU Delft	$\frac{\partial G_N(\|\mathbf{x}-\mathbf{x_0}\|)}{\partial n} = 0$ Rayleigh I for planar SSD modern WFS [Spo08a]

Dirichlet Boundary Condition on Sound Pressure

Equivalent sound-soft scattering [Faz13, Zot13]

$$\alpha\, P(\mathbf{x}) = \oint_{\partial V} \left[-\frac{\partial S_T(\mathbf{x_0})}{\partial n}\, G(|\mathbf{x}-\mathbf{x_0}|) + \frac{\partial G(|\mathbf{x}-\mathbf{x_0}|)}{\partial n}\, S_T(\mathbf{x_0}) \right] \mathrm{d}A(\mathbf{x_0})$$

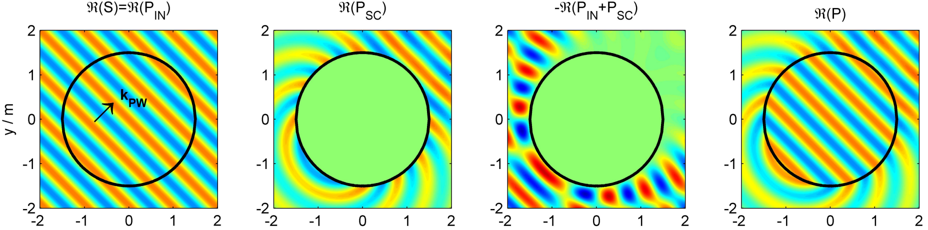
- desired sound field $S(\mathbf{x}) = P_{\mathsf{IN}}(\mathbf{x})$ is modeled as an incident field for a scatterer
- 'total' sound field $S_T(\mathbf{x}) = P_{\mathsf{IN}}(\mathbf{x}) + P_{\mathsf{SC}}(\mathbf{x})$ for $\mathbf{x} \notin V$
- impose Dirichlet boundary condition $S_T(\mathbf{x_0}) = 0$, i.e. ideal sound-soft scatterer
- scattered field $P_{\mathsf{SC}}(\mathbf{x})$ has to be known :-(

$$\oint_{\partial V} \underbrace{\left[-\frac{\partial P_{\mathsf{IN}}(\mathbf{x_0})}{\partial n} - \frac{\partial P_{\mathsf{SC}}(\mathbf{x_0})}{\partial n} \right]}_{D(\mathbf{x_0})} G(|\mathbf{x}-\mathbf{x_0}|)\mathrm{d}A(\mathbf{x_0}) = \begin{cases} +P_{\mathsf{IN}}(\mathbf{x}), & \mathbf{x} \in V \\ +P_{\mathsf{IN}}(\mathbf{x}), & \mathbf{x} \in \partial V \\ -P_{\mathsf{SC}}(\mathbf{x}), & \mathbf{x} \notin V \end{cases}$$

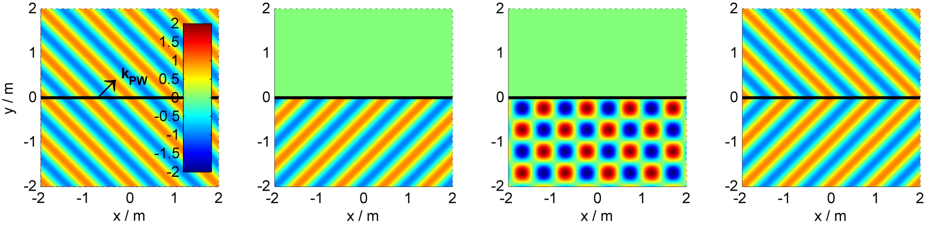
Dirichlet Boundary Condition on Sound Pressure

Equivalent sound-soft scattering [Faz13, Zot13]

- spherical secondary source distribution



- planar secondary source distribution



Neumann Boundary Condition on Green's Function

Rayleigh integral equation $\longrightarrow$ starting point for modern WFS formulation [Spo08a]

$$\alpha\, P(\mathbf{x}) = \oint\limits_{\partial V} \left[-\frac{\partial S(\mathbf{x_0})}{\partial n} G_N(|\mathbf{x}-\mathbf{x_0}|) + \frac{\partial G_N(|\mathbf{x}-\mathbf{x_0}|)}{\partial n} S(\mathbf{x_0}) \right] \mathrm{d}A(\mathbf{x_0})$$

- **special** case for the **planar** secondary source distribution

$$\frac{\partial G_N(|\mathbf{x}-\mathbf{x_0}|)}{\partial n} = 0 \qquad G_N(|\mathbf{x}-\mathbf{x_0}|) = 2 \cdot G(|\mathbf{x}-\mathbf{x_0}|)$$

$$P(\mathbf{x}) = \iint\limits_{-\infty}^{+\infty} \underbrace{-\frac{\partial S(\mathbf{x_0})}{\partial n} \cdot 2}_{D(\mathbf{x_0})} \cdot G(|\mathbf{x}-\mathbf{x_0}|)\, \mathrm{d}x \mathrm{d}z$$

- well known link between Rayleigh integral and angular spectrum propagator with Weyl integral representation of $G(|\mathbf{x}-\mathbf{x_0}|)$ [Faz10, Faz13]

Equivalent Solutions for the Single Layer Potential

In conclusion calculus leads to equivalent driving functions by employing

- for a spherical secondary source distribution

mathematical approach: **spectral representation**

$\stackrel{!}{=}$ mode matching →
$\stackrel{!}{=}$ convolution over a sphere →

$D(\mathbf{x_0})$

physical approach: **Helmholtz integral**

← Dirichlet on $S_T(\mathbf{x})$

- for a planar secondary source distribution

mathematical approach: **spectral representation**

$\stackrel{!}{=}$ mode matching →
$\stackrel{!}{=}$ convolution over a plane →

$D(\mathbf{x_0})$

physical approach: **Helmholtz integral**

← Dirichlet on $S_T(\mathbf{x})$
← Neumann on $G(|\mathbf{x} - \mathbf{x_0}|)$

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[Ahr08] Ahrens, J.; Spors, S. (2008): "An analytical approach to sound field reproduction using circular and spherical loudspeaker distributions." In: *Acta Acust United Ac*, **94**(6):988–999.

[Ahr10] Ahrens, J.; Spors, S. (2010): "Sound field reproduction using planar and linear arrays of loudspeakers." In: *IEEE Trans. Audio Speech Language Process.*, **18**(8):2038–2050.

[Faz10] Fazi, F.M. (2010): *Sound Field Reproduction*. Ph.D. thesis, University of Southampton, Institute of Sound and Vibration Research.

[Faz12] Fazi, F.M.; Nelson, P.A. (2012): "Nonuniqueness of the solution of the sound field reproduction problem with boundary pressure control." In: *Acta Acust United Ac*, **98**(1):1–14.

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[Spo08a] Spors, S.; Rabenstein, R.; Ahrens, J. (2008): "The theory of Wave Field Synthesis revisited." In: *Proc. of the 124th Audio Eng. Soc. Convention, Amsterdam*, #7358.

[Spo08b] Spors, S.; Rath, M.; Ahrens, J. (2008): "Towards a theory for arbitrarily shaped sound field reproduction systems." In: *Abstract and Talk for Acoustics'08, Paris*, #2991.

[Zot13] Zotter, F.; Spors, S. (2013): "Is sound field control determined at all frequencies? How is it related to numerical acoustics?" In: *Proc. of the 52nd Audio Eng. Soc. Intl. Conference on Sound Field Control, Guildford*, .

Thanks for your attention

slides of this talk avalaible @

`http://spatialaudio.net/`

Schultz, F.; Spors, S. (2014): "Comparing Approaches to the Spherical and Planar Single Layer Potentials for Interior Sound Field Synthesis." In: *Proc. of the EAA Joint Symposium on Auralization and Ambisonics 2014, Berlin*.

Fredholm Integral in Time Signal Processing

$$y(t) = \int_{-\infty}^{+\infty} x(\tau) \cdot h(t-\tau)\,\mathrm{d}\tau \quad \circ\!\!-\!\!\bullet \quad Y(\omega) = X(\omega) \cdot H(\omega)$$

- **infinite** and **nondenumerable** eigensignals and corresponding signal-/system eigenvalues

$\longrightarrow$ **continuous** spectrum of eigenvalues/singular values

- for this special case solution via Fourier transform available

$$h(t-\tau) = \frac{1}{2\,\pi} \int_{-\infty}^{+\infty} H(\omega)\, \mathrm{e}^{+\mathrm{j}\,\omega\,t}\, \mathrm{e}^{-\mathrm{j}\,\omega\,\tau}\, \mathrm{d}\omega$$

$$\begin{Bmatrix} y(t) \\ x(t) \end{Bmatrix} = \frac{1}{2\,\pi} \int_{-\infty}^{+\infty} \begin{Bmatrix} Y(\omega) \\ X(\omega) \end{Bmatrix} \mathrm{e}^{+\mathrm{j}\,\omega\,t}\, \mathrm{d}\omega$$

$$x(\tau) = \frac{1}{2\,\pi} \int_{-\infty}^{+\infty} \underbrace{\frac{Y(\omega)}{H(\omega)}}_{X(\omega)} \mathrm{e}^{+\mathrm{j}\,\omega\,\tau}\, \mathrm{d}\omega$$

Convolution Integral for Time Signal Processing

- the convolution integral for time-domain signal processing is a **Fredholm integral** of first kind

$$y(t) = \int_{-\infty}^{+\infty} x(\tau) \cdot h(t-\tau)\, \mathrm{d}\tau \quad \circ\!\!-\!\!\bullet \quad Y(\omega) = X(\omega) \cdot H(\omega)$$

- system identification $\rightarrow$ heavily used to obtain LTI-system impulse responses

$$H(\omega) = \frac{Y(\omega)}{X(\omega)}$$

- signal identification $\rightarrow$ link to the inverse problem of sound field synthesis

$$X(\omega) = \frac{Y(\omega)}{H(\omega)} \quad \rightarrow \quad x(\tau) = \int_{-\infty}^{+\infty} \frac{Y(\omega)}{H(\omega)}\, \mathrm{e}^{+\mathrm{j}\,\omega\,\tau}\, \mathrm{d}\omega$$

- links between the time domain convolution and the single layer potential

$$x(\tau) \leftrightarrow D(\mathbf{x_0}) \qquad h(t-\tau) \leftrightarrow G(|\mathbf{x}-\mathbf{x_0}|) \qquad y(t) \leftrightarrow P(\mathbf{x})$$

Spherical Harmonics Transform

- orthonormal & completeness relation of spherical surface harmonics

$$\int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} Y_n^m(\theta,\phi)\; Y_{n'}^{m'}(\theta,\phi)^* \;\sin(\theta)\,\mathrm{d}\theta\,\mathrm{d}\phi = \delta_{nn'}\delta_{mm'}$$

$$\sum_{n=0}^{+\infty} \sum_{m=-n}^{+m} Y_n^m(\theta,\phi)\; Y_n^m(\theta_0,\phi_0)^* = \delta(\phi-\phi_0)\,\delta(\cos\theta - \cos\theta_0)$$

- expansion (inverse $\mathcal{SHT}$) & decomposition ($\mathcal{SHT}$) for $|\mathbf{x}| =$ const

$$P(\mathbf{x}) = \sum_{n=0}^{+\infty} \sum_{m=-n}^{+n} P_n^m(r)\; Y_n^m(\theta,\phi)$$

$$P_n^m(r) = \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} P(\mathbf{x})\; Y_n^m(\theta,\phi)^* \;\sin\theta\,\mathrm{d}\theta\,\mathrm{d}\phi$$

Solving the Fredholm Integral for Spherical SSD

Ambisonics solution for a virtual **point** source at $\mathbf{x}_{\text{PS}}$, $\mathrm{e}^{+\mathrm{j}\,\omega\,t}$-convention

$$G(|\mathbf{x}-\mathbf{x_0}|) = \sum_{n=0}^{\infty} \sum_{m=-n}^{+n} -\mathrm{j}\,\frac{\omega}{c}\,h_n^{(2)}(\frac{\omega}{c}\,r_0)\,j_n(\frac{\omega}{c}\,r)\,Y_n^m(\theta,\phi)\;Y_n^m(\theta_0,\phi_0)^*$$

$$S(\mathbf{x}) = \sum_{n=0}^{\infty} \sum_{m=-n}^{+n} \underbrace{\overbrace{-\mathrm{j}\,\frac{\omega}{c}\,h_n^{(2)}(\frac{\omega}{c}\,r_{\text{PS}})\;Y_n^m(\theta_{\text{PS}},\phi_{\text{PS}})^*}^{b_{nm}}\;j_n(\frac{\omega}{c}\,r)}_{S_n^m(r)}\;Y_n^m(\theta,\phi)$$

$$D(\mathbf{x_0}) = \sum_{n=0}^{\infty} \sum_{m=-n}^{+n} \frac{1}{r_0^2}\,\frac{S_n^m(r)}{\underbrace{-\mathrm{j}\,\frac{\omega}{c}\,h_n^{(2)}(\frac{\omega}{c}\,r_0)\,j_n(\frac{\omega}{c}\,r)}_{D_n^m(r)}}\;Y_n^m(\theta_0,\phi_0)$$

$$D(\mathbf{x_0}) = \sum_{n=0}^{\infty} \sum_{m=-n}^{+n} \frac{1}{r_0^2}\,\frac{-\mathrm{j}\,\frac{\omega}{c}\,h_n^{(2)}(\frac{\omega}{c}\,r_{\text{PS}})\;Y_n^m(\theta_{\text{PS}},\phi_{\text{PS}})^*\,j_n(\frac{\omega}{c}\,r)}{-\mathrm{j}\,\frac{\omega}{c}\,h_n^{(2)}(\frac{\omega}{c}\,r_0)\,j_n(\frac{\omega}{c}\,r)}\;Y_n^m(\theta_0,\phi_0)$$

Solving the Fredholm Integral for Spherical SSD

Ambisonics solution for a virtual **plane** wave into $\mathbf{k}_{\text{PW}}$, $\mathrm{e}^{+\mathrm{j}\,\omega\,t}$-convention

$$G(|\mathbf{x}-\mathbf{x_0}|) = \sum_{n=0}^{\infty}\sum_{m=-n}^{+n} -\mathrm{j}\,\frac{\omega}{c}\,h_n^{(2)}(\frac{\omega}{c}\,r_0)\,j_n(\frac{\omega}{c}\,r)\,Y_n^m(\theta,\phi)\;Y_n^m(\theta_0,\phi_0)^*$$

$$S(\mathbf{x}) = \sum_{n=0}^{\infty}\sum_{m=-n}^{+n} \underbrace{\overbrace{4\,\pi\,\mathrm{j}^{-n}\;Y_n^m(\theta_{\text{PW}},\phi_{\text{PW}})^*}^{b_{nm}}\,j_n\left(\frac{\omega}{c}\,r\right)}_{S_n^m(r)}\;Y_n^m(\theta,\phi)$$

$$D(\mathbf{x_0}) = \sum_{n=0}^{\infty}\sum_{m=-n}^{+n} \frac{1}{r_0^2}\,\frac{S_n^m(r)}{\underbrace{-\mathrm{j}\,\frac{\omega}{c}\,h_n^{(2)}(\frac{\omega}{c}\,r_0)\,j_n(\frac{\omega}{c}\,r)}_{D_n^m(r)}}\;Y_n^m(\theta_0,\phi_0)$$

$$D(\mathbf{x_0}) = \sum_{n=0}^{\infty}\sum_{m=-n}^{+n} \frac{1}{r_0^2}\,\frac{4\,\pi\,\mathrm{j}^{-n}\;Y_n^m(\theta_{\text{PW}},\phi_{\text{PW}})^*\,j_n\left(\frac{\omega}{c}\,r\right)}{-\mathrm{j}\,\frac{\omega}{c}\,h_n^{(2)}(\frac{\omega}{c}\,r_0)\,j_n(\frac{\omega}{c}\,r)}\;Y_n^m(\theta_0,\phi_0)$$

Spatial Fourier Transform

- δ-normalized relation

$$\iint\limits_{-\infty}^{+\infty} \frac{\mathrm{e}^{-\mathrm{j}(k_x' x + k_z' z)}}{2\,\pi} \, \frac{\mathrm{e}^{+\mathrm{j}(k_x x + k_z z)}}{2\,\pi} \mathrm{d}x \mathrm{d}z = \delta(k_x - k_x')\,\delta(k_z - k_z')$$

$$\iint\limits_{-\infty}^{+\infty} \frac{\mathrm{e}^{+\mathrm{j}(k_x x' + k_z z')}}{2\,\pi} \, \frac{\mathrm{e}^{-\mathrm{j}(k_x x + k_z z)}}{2\,\pi} \mathrm{d}k_x \mathrm{d}k_z = \delta(x - x')\,\delta(z - z')$$

- expansion (inverse $\mathcal{F}$) & decomposition ($\mathcal{F}$) for $y > 0$ & $y =$ const

$$P(\mathbf{x}) = \frac{1}{2\,\pi} \iint\limits_{-\infty}^{+\infty} P(k_x, y, k_z)\, \mathrm{e}^{-\mathrm{j}\,(k_x\,x + k_z\,z)} \mathrm{d}k_x\, \mathrm{d}k_z$$

$$P(k_x, y, k_z) = \frac{1}{2\,\pi} \iint\limits_{-\infty}^{+\infty} P(\mathbf{x})\, \mathrm{e}^{+\mathrm{j}\,(k_x\,x + k_z\,z)} \mathrm{d}x\, \mathrm{d}z$$

Possible Wave Numbers

- for wave radiation into target halfspace $y > 0$

$$k_y = \begin{cases} +\sqrt{(\frac{\omega}{c})^2 - k_x^2 - k_z^2} & \text{for } (\frac{\omega}{c})^2 \geq k_x^2 - k_z^2 \qquad \text{propagating} \\ -\mathrm{j}\sqrt{k_x^2 + k_z^2 - (\frac{\omega}{c})^2} & \text{for } k_x^2 - k_z^2 > (\frac{\omega}{c})^2 \qquad \text{evanescent} \end{cases}$$

Solving the Fredholm Integral for Planar SSD

Spectral Division Method solution for a **plane** wave into $\mathbf{k}_{\text{PW}}$, $\mathrm{e}^{+\mathrm{j}\,\omega\,t}$-convention, $y > 0$

$$G(|\mathbf{x}-\mathbf{x_0}|) = \frac{1}{(2\,\pi)^2} \iint\limits_{-\infty}^{+\infty} -\mathrm{j}\,\frac{\mathrm{e}^{-\mathrm{j}\,k_y \cdot y}}{2\,k_y}\,\mathrm{e}^{-\mathrm{j}(k_x\,x + k_z\,z)}\,\mathrm{e}^{+\mathrm{j}(k_x\,x_0 + k_z\,z_0)} \mathrm{d}k_x\,\mathrm{d}k_z$$

$$S(\mathbf{x}) = \frac{1}{(2\,\pi)^2} \iint\limits_{-\infty}^{+\infty} S(k_x, y, k_z)\,\mathrm{e}^{-\mathrm{j}\,(k_x\,x + k_z\,z)} \mathrm{d}k_x\,\mathrm{d}k_z$$

$$S(k_x, y, k_z) = 4\,\pi^2\,\delta(k_x - k_{x,\text{PW}})\,\delta(k_z - k_{z,\text{PW}})\,\mathrm{e}^{-\mathrm{j}\,k_{y,\text{PW}} \cdot y}$$

$$D(\mathbf{x_0}) = \frac{1}{(2\,\pi)^2} \iint\limits_{-\infty}^{+\infty} \underbrace{\frac{S(k_x, y, k_z)}{-\mathrm{j}\,\frac{\mathrm{e}^{-\mathrm{j}\,k_y \cdot y}}{2\,k_y}}}_{D(k_x, y, k_z)}\,\mathrm{e}^{-\mathrm{j}\,(k_x\,x_0 + k_z\,z_0)} \mathrm{d}k_x\,\mathrm{d}k_z$$

$$D(k_x, k_z) = 8\,\pi^2\,\mathrm{j}\,k_{y,\text{PW}} \cdot \delta(k_x - k_{x,\text{PW}}) \cdot \delta(k_z - k_{z,\text{PW}})$$

$$S(k_x, y, k_z) = D(k_x, k_z) \cdot G(k_x, y, k_z)$$

Equivalent Sound-Soft Scattering for Planar SSD I

$$P_{\mathsf{IN}}(k_x, y, k_z) = \check{P}_{\mathsf{IN}}(k_x, k_z)\,\mathrm{e}^{-\mathrm{j}\,k_y\,y}$$

$$P_{\mathsf{SC}}(k_x, y, k_z) = \check{P}_{\mathsf{SC}}(k_x, k_z)\,\mathrm{e}^{+\mathrm{j}\,k_y\,y}$$

$$\left[P_{\mathsf{IN}}(k_x, y, k_z) + P_{\mathsf{SC}}(k_x, y, k_z)\right]\Big|_{y=0} = 0$$

$$\check{P}_{\mathsf{SC}}(k_x, k_z) = -\check{P}_{\mathsf{IN}}(k_x, k_z)$$

$$P_{\mathsf{IN}}(\mathbf{x}) = \frac{1}{4\,\pi^2}\iint\limits_{-\infty}^{+\infty} \check{P}_{\mathsf{IN}}(k_x, k_z)\,\mathrm{e}^{-\mathrm{j}\,k_y\,y}\,\mathrm{e}^{-\mathrm{j}\,[k_x\,x+k_z\,z]}\mathrm{d}k_x\mathrm{d}k_z$$

$$P_{\mathsf{SC}}(\mathbf{x}) = \frac{1}{4\,\pi^2}\iint\limits_{-\infty}^{+\infty} \check{P}_{\mathsf{SC}}(k_x, k_z)\,\mathrm{e}^{+\mathrm{j}\,k_y\,y}\,\mathrm{e}^{-\mathrm{j}\,[k_x\,x+k_z\,z]}\mathrm{d}k_x\mathrm{d}k_z$$

$$D(\mathbf{x_0}) = -\frac{\partial P_{\mathsf{SC}}(\mathbf{x_0})}{\partial n} - \frac{\partial P_{\mathsf{IN}}(\mathbf{x_0})}{\partial n}$$

Equivalent Sound-Soft Scattering for Planar SSD II

$$D(\mathbf{x_0}) = \mathrm{j}\, k_y \, \frac{1}{4\,\pi^2} \iint\limits_{-\infty}^{+\infty} 2 \cdot \check{P}_{\mathrm{IN}}(k_x, k_z)\, \mathrm{e}^{-\mathrm{j}\,(k_x x_0 + k_z z_0)} \mathrm{d}k_x \, \mathrm{d}k_z$$

- link to SDM by expanding with $\mathrm{e}^{-\mathrm{j}\, k_y\, y} / \mathrm{e}^{-\mathrm{j}\, k_y\, y}$ and rearranging to

$$D(\mathbf{x_0}) = \frac{1}{4\,\pi^2} \iint\limits_{-\infty}^{+\infty} \frac{\overbrace{\check{P}_{\mathrm{IN}}(k_x, k_z)\, \mathrm{e}^{-\mathrm{j}\, k_y\, y}}^{S(k_x, y, k_z)}}{-\mathrm{j}\, \frac{\mathrm{e}^{-\mathrm{j}\, k_y\, y}}{2\, k_y}} \mathrm{e}^{-\mathrm{j}\,(k_x x_0 + k_z z_0)} \mathrm{d}k_x \, \mathrm{d}k_z$$

- link to Rayleigh integral equation: term within brackets is $-2 \cdot S(\mathbf{x_0})$

$$D(\mathbf{x_0}) = \frac{\partial \left[\frac{-2}{4\,\pi^2} \iint\limits_{-\infty}^{+\infty} \overbrace{\check{P}_{\mathrm{IN}}(k_x, k_z)\, \mathrm{e}^{-\mathrm{j}\, k_y\, y}}^{S(k_x, y, k_z)}\, \mathrm{e}^{-\mathrm{j}\,(k_x\, x_0 + k_z\, z_0)} \mathrm{d}k_x \mathrm{d}k_z \right]}{\partial y} \Bigg|_{y=0}$$