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Wave Field Synthesis (WFS)

- an analytic method for sound field synthesis [Berkhout et al., 1993, Spors et al., 2008]

$$S(x, \omega) = \int_{\partial V_0} D(x_0, \omega) G(x - x_0, \omega) \, dA, \quad x \in V_0$$

- based on the Rayleigh integral equation

$$S(x, \omega) = \int_{-\infty}^{\infty} -2 \frac{\partial}{\partial n_0} S(x_0, \omega) G(x - x_0, \omega) \, d\chi_0 \, dz_0, \quad y > y_0$$

- analytic solution for linear/planar arrays
- approximated solution for arbitrary convex-shaped arrays (high-frequency approximation)
Synthesis of a Virtual Plane Wave

2.5D Driving Function [Spors et al., 2008, Schultz, 2016]

Frequency domain
\[ D(x, \omega) = \sqrt{i \frac{\omega}{c}} a(x_0) \sqrt{8\pi} e^{-i \frac{\omega}{c} \langle n_{pw}, n_0(x_0) \rangle} \]

Time domain
\[ d(x, t) = h_{pre}(t) \ast a(x_0) \sqrt{8\pi} e^{-i \frac{\omega}{c} \langle n_{pw}, n_0(x_0) \rangle} \delta \left( t - \frac{\langle n_{pw}, x_0 \rangle}{c} \right) \]

\( n_0 \) \hspace{1cm} \text{inward pointing normal vector}

\( n_{pw} \) \hspace{1cm} \text{plane wave direction}

\( \phi_n = \arccos(\langle n_{pw}, n_0 \rangle) \)

\( a(x_0) \) \hspace{1cm} \{1\} \text{ or } \{0\}
Spatial Aliasing and Local WFS

- Using a finite number of loudspeakers results in spatial aliasing artifacts ($f > f_{al}$)
- Local WFS achieves increased accuracy within a local area
  1. Virtual loudspeaker array [Spors and Ahrens, 2010]
  2. Virtual equivalent scattering and time reversal [Spors et al., 2011]
  3. Spatial band-limitation [Hahn et al., 2016]

[circular array $R = 1.5$, 56 loudspeakers, $\phi_{pw} = -90^\circ$]
using a finite number of loudspeakers results in spatial aliasing artifacts ($f > f_{\text{al}}$)

- local WFS achieves increased accuracy within a local area
  1. virtual loudspeaker array [Spors and Ahrens, 2010]
  2. virtual equivalent scattering and time reversal [Spors et al., 2011]
  3. spatial band-limitation [Hahn et al., 2016]
Local WFS by Spatial Band-Limitation in the Circular Harmonics Domain

[Hahn et al., 2016]

- circular harmonics expansion of a plane wave with respect to $x_c$

\[ e^{-i \frac{\omega}{c} \langle n_{pw}, x \rangle} = e^{-i \frac{\omega}{c} \langle n_{pw}, x_c \rangle} \sum_{m=-\infty}^{\infty} i^{-m} e^{-i m \phi_{pw}} J_m \left( \frac{\omega}{c} r' \right) e^{i m \phi'} , \quad \text{where} \quad x - x_c = (r', \phi', z') \]

**spatial band-limitation:** $M \Rightarrow$ describes the sound field within a circular region $r_M \approx \frac{c M}{2 \pi f}$ [Kennedy et al., 2007]

- WFS driving function based on spatially band-limited circular harmonics expansion
- driving function available only in the frequency domain so far 😞
Local WFS by Spatial Band-Limitation in the Circular Harmonics Domain

[Hahn et al., 2016]

- circular harmonics expansion of a plane wave with respect to $x_c$

$$e^{-i\frac{\omega_c}{c} \langle n_{pw},x \rangle} \approx e^{-i\frac{\omega_c}{c} \langle n_{pw},x_c \rangle} \sum_{m=-M}^{M} i^{-m} e^{-im\phi_{pw}} J_m\left(\frac{\omega_c}{c} r'\right) e^{im\phi'},$$

where $x - x_c = (r', \phi', z')$

**spatial band-limitation**: $M \Rightarrow$ describes the sound field within a circular region $r_M \approx \frac{c M}{2\pi f}$ [Kennedy et al., 2007]

- WFS driving function based on spatially band-limited circular harmonics expansion

- driving function available only in the frequency domain so far 😐
Circular Harmonics Expansion ($x_c = 0$)

\[ S(x, \omega) = \sum_{m=-M}^{M} i^{-m} e^{-im\phi_{pw}} J_m(\frac{\omega c r}{r}) e^{im\phi} \]

Plane Wave Decomposition

\[ S(x, \omega) = \frac{1}{2\pi} \int_{0}^{2\pi} D_M(\alpha - \phi_{pw}) e^{-i \frac{\omega}{c} r \cos(\phi - \alpha)} d\alpha \]

where \( D_M(\alpha - \phi_{pw}) \equiv \sin \left( \frac{2M+1}{2} (\alpha - \phi_{pw}) \right) / \sin \left( \frac{1}{2} (\alpha - \phi_{pw}) \right) \)

\[ F_{t}^{-1} \]

\[ s(x, t) = \frac{1}{2\pi} \int_{0}^{2\pi} D_M(\alpha - \phi_{pw}) \delta \left( t - \frac{r}{c} \cos(\phi - \alpha) \right) d\alpha \]

WFS Driving Function

\[ d_{PWD}(x_0, t) = \sqrt{\frac{2r}{\pi}} \int_{0}^{2\pi} a(x_0, \alpha) \langle n_{\alpha}, n_0 \rangle \times D_M(\alpha - \phi_{pw}) \delta \left( t - \frac{r}{c} \cos(\phi - \alpha) \right) d\alpha \]
Time-Domain Representation II

\[
s(x, t) = \frac{1}{2\pi} \int_{0}^{2\pi} D_{M}(\alpha - \phi_{pw}) \delta \left( t - \frac{r}{c} \cos(\phi - \alpha) \right) d\alpha
\]

\[
s(x, t) = \frac{1}{2\pi} \frac{\Pi \left( \frac{ct}{2r} \right)}{|\sin \beta(t)|} \left[ D_{M}(\phi - \phi_{pw} - \pi + \beta(t)) \delta \left( t - \frac{r}{c} \cos(\beta(t) + \pi) \right) + D_{M}(\phi - \phi_{pw} - \pi - \beta(t)) \delta \left( t - \frac{r}{c} \cos(-\beta(t) + \pi) \right) \right]
\]

where \( \beta(t) = \arccos \left( -\frac{ct}{r} \right) \) and \( \Pi(\cdot) \): rectangular function

Circular Harmonics Expansion

- two plane waves at each time instance for \(-\frac{r}{c} < t < \frac{r}{c}\)
  \[
  \phi_{pw}^{+}(t) = \phi + \beta(t) + \pi \\
  \phi_{pw}^{-}(t) = \phi - \beta(t) + \pi
  \]
Spatially Band-Limited Plane Wave

\[ \phi_{pw}^+(t) = \phi + \beta(t) + \pi \]

\[ \phi_{pw}^-(t) = \phi - \beta(t) + \pi \]

original plane wave
WFS Driving Function

\[
d(x_0, t) = \sqrt{\frac{2r}{\pi}} \frac{\Pi(\frac{ct}{2r})}{|\sin \beta(t)|} \times \left[ a(x_0, \phi_{pw}^+(t)) \langle n_{pw}^+(t), n_0 \rangle \mathcal{D}_M (\phi_{pw}^+(t)) + a(x_0, \phi_{pw}^-(t)) \langle n_{pw}^-(t), n_0 \rangle \mathcal{D}_M (\phi_{pw}^-(t)) \right]
\]

\[
x_c = (0, 0, 0), x_0 = (1.5, 0, 0), \phi_{pw} = -\frac{\pi}{3}, \phi_n = -\frac{3\pi}{4}
\]
Practical Implementation

PWC

\[ d_{\text{PWC}}(x_0, t) = \sqrt{\frac{2r}{\pi}} \int_0^{2\pi} a(x_0, \alpha) \langle n_\alpha, n_0 \rangle \times D_M (\alpha - \phi_{pw}) \delta \left( t - \frac{r}{c} \cos(\phi - \alpha) \right) d\alpha \]

CHT

\[ d_{\text{CHT}}(x_0, t) = \sqrt{\frac{2r}{\pi}} \frac{n(t)}{|\sin \beta(t)|} \times \left[ a(x_0, \phi_{pw}^+(t)) \langle n_{pw}^+(t), n_0 \rangle D_M (\phi_{pw}^+(t)) + a(x_0, \phi_{pw}^-(t)) \langle n_{pw}^-(t), n_0 \rangle D_M (\phi_{pw}^-(t)) \right] \]

- temporal sampling: integer or fractional delay
- spatial sampling: synthesize only a finite number of plane waves e.g. equi-angular sampling
- temporal sampling implicitly discretizes the plane wave angles constitutes a nonuniform sampling of \( \phi_{pw}^\pm \)
- anti-aliasing filtering and dynamic range control using \( \arctan(\cdot) \)

\[ \mathcal{L}(u) = \frac{2A_{\text{thr}}}{\pi} \arctan \left( \frac{\pi u}{2A_{\text{thr}}} \right) \]
Practical Implementation

**PWD**

\[
d_{\text{PWD}}(x_0, t) = \sqrt{\frac{2r}{\pi}} \pi \int_0^{2\pi} a(x_0, \alpha) \langle n_\alpha, n_0 \rangle \times D_M (\alpha - \phi_{pw}) \delta \left( t - \frac{r}{c} \cos(\phi - \alpha) \right) d\alpha
\]

**CHT**

\[
d_{\text{CHT}}(x_0, t) = \sqrt{\frac{2r}{\pi}} \pi \int \frac{\Pi \left( \frac{ct}{2r} \right)}{| \sin \beta(t) |} \times \left[ a \left( x_0, \phi_{pw}^+(t) \right) \langle n_{pw}^+(t), n_0 \rangle D_M \left( \phi_{pw}^+(t) \right) + a \left( x_0, \phi_{pw}^-(t) \right) \langle n_{pw}^-(t), n_0 \rangle D_M \left( \phi_{pw}^-(t) \right) \right]
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\]
### 2.5D WFS Simulation

#### Configuration
- **circular array** \( R = 1.5 \text{ m}, \ N = 60 \) loudspeakers
- **virtual plane wave** \( \phi_{pw} = -90^\circ \) (downwards)
- **spatial bandwidth** \( M = 29 \)
- **sampling frequency** \( f_s = 44.1 \text{ kHz} \)

#### Driving Functions
- **conventional WFS**
- **circular harmonics expansion (CHT)**
- **plane wave decomposition using integer delay (PWD-i)**
- **plane wave decomposition using fractional delay (PWD-f)**
  20th-order Lagrange polynomial

† number of plane waves \( N_{pw} = 385 \) matched for CHT, PWD-i, and PWD-f
**Driving Function and Synthesized Sound Field**

- **Conventional WFS**
  - driving function given as delay and weight
  - straight wavefront followed by spatial aliasing components

- **Spatially band-limited WFS**
  - peaks of the driving function always on the v-shaped contour
  - detailed structure depends on the expansion center of the circular harmonics expansion
  - improved accuracy around $x_C$
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Temporal Properties

Synthesized Sound Field (CHT)

Synthesized Sound Field (PWD-f)

Synthesized Sound Field (PWD-i)

Impulse Responses

PWD-i

PWD-f

CHT


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Spectral Properties

- **CHT**: optimal $A_{\text{thr}} = 17$ found empirically
- **CHT**: high-frequency deviations ($< 3$ dB) due to temporal aliasing
- **PWD-i** & **PWD-f**: fluctuations at high frequencies
- **Typical low-frequency deviation in WFS** due to high-frequency approximation
Improve by Oversampling $\uparrow 2$

- CHT: $A_{thr} = 24$
- $f_s = 88.2$ kHz, $N_{pw} = 771$
- CHT: outperforms PWD-i and comparable to PWD-f
- PWD-f: computational complexity increases significantly
Conclusion

Summary
- two time-domain representations introduced (CHT & PWD) for a spatially band-limited plane
- corresponding WFS driving functions derived
- improved accuracy achieved within a freely movable local listening area
- CHT, PWD-i, and PWD-f compared in terms of temporal/spectral/spatial properties

Future Work
- CHT: improved anti-aliasing filtering and dynamic range control
- physical/perceptual comparison with other local WFS methods

What about POINT SOURCES?
⇒ Fiete Winter et al., "Time-Domain Realisation of Model-Based Rendering for 2.5D Local Wave Field Synthesis Using Spatial Bandwidth-Limitation" [AASP-L3, 11:00, Aegle B]
References I


Local Sound Field Synthesis by Virtual Acoustic Scattering and Time-reversal.

The Theory of Wave Field Synthesis Revisited.
Circular Harmonics Expansion of a Plane Wave

- plane wave with angle $\phi_{pw}$ and wave vector $k_{pw} = [\frac{\omega}{c} \cos \phi_{pw}, \frac{\omega}{c} \sin \phi_{pw}, 0]^T$ (parallel to the $xy$-plane)
- $x_c$: expansion center
- approximated with a final number of terms $M$

$$e^{-i\langle k_{pw}, x \rangle} = \sum_{m=-\infty}^{\infty} i^{-m} e^{im\phi_{pw}} \frac{J_m(\omega c r)}{J_m(\omega c r') e^{im\phi}}$$

Expansion coefficient Bessel function Circular harmonics

$$= e^{-i\langle k_{pw}, x_c \rangle} \sum_{m=-\infty}^{\infty} i^{-m} e^{im\phi_{pw}} \frac{J_m(\omega c r')}{e^{im\phi}} \quad x = (r, \phi, z)$$

$\quad x - x_c = (r', \phi', z')$ spatial shift

$$\approx e^{-i\langle k_{pw}, x_c \rangle} \sum_{m=-M}^{M} i^{-m} e^{im\phi_{pw}} \frac{J_m(\omega c r')}{e^{im\phi'}} \quad \text{spatial band-limitation}$$
Inverse Fourier Transform of the Bessel Functions

- circular harmonics expansion

\[ S_{pw}^{(M)}(x, \omega) = e^{-i \frac{\omega}{c} \langle n_{pw}, x_c \rangle} \sum_{m=-M}^{M} i^{-m} e^{-im\phi_{pw}} J_m(\frac{\omega}{c} r') e^{im\phi'} \]

- forward Fourier transform of the Bessel function [Abramowitz and Stegun, 1964]

\[ \mathcal{F}\{J_m(t)\} = \int_{-\infty}^{\infty} J_m(t) e^{-i\omega t} \, dt = \frac{2(-i)^m \Pi(\omega) T_m(\omega)}{\sqrt{1 - \omega^2}} \]

- exploiting the duality of Fourier transform, the inverse Fourier transform of \( J_m(\frac{\omega}{c} r) \) reads

\[ \mathcal{F}^{-1}\{J_m(\frac{\omega}{c} r)\} = \frac{i^m}{\pi} \frac{\Pi(\frac{ct}{2r}) T_m(\frac{ct}{r})}{\sqrt{\left(\frac{t}{c}\right)^2 - t^2}} \]

\( T_m(\cdot) \) Chebyshev polynomial of the first kind of degree \( m \)

\( \Pi(\cdot) \) rectangular function
Time Domain Representations

\[ S(x, \omega) = \sum_{m=-M}^{M} i^{-m} e^{-im \phi_{pw}} J_m \left( \frac{\omega}{c} r' \right) e^{im \phi'} \]

Plane Wave Decomposition

\[ S(x, \omega) = \frac{1}{2\pi} \int_{0}^{2\pi} \mathcal{D}_M(\alpha - \phi_{pw}) e^{-i \frac{\omega}{c} \cos(\phi - \alpha)} d\alpha \]

\[ \mathcal{F}_t^{-1} \]

sifting property of \( \delta(\cdot) \)

\[ s(x, t) = \frac{1}{2\pi} \int_{0}^{2\pi} \mathcal{D}_M(\alpha - \phi_{pw}) \delta \left( t - \frac{r}{c} \cos(\phi - \alpha) \right) d\alpha \]

\[ \mathcal{F}_t^{-1} \]

\[ s(x, t) = \frac{1}{2\pi} \frac{\Pi \left( \frac{ct}{2r} \right)}{\left| \sin \beta(t) \right|} \left[ \mathcal{D}_M(\phi - \phi_{pw} - \pi + \beta(t)) \right. \]
\[ \left. + \mathcal{D}_M(\phi - \phi_{pw} - \pi - \beta(t)) \right] \]
Distribution of Plane Wave Directions

- **PWD**: uniform distribution
- **CHT**: non-uniform distribution

\[ x_c = (0.5, 0, 0), \phi_{pw} = -90^\circ, N_{pw} = 258 \]
Dynamic Range Control

only for CHT

\[ A_{\text{thr}} = 100 \]

- \[ \frac{1}{|\sin \beta(t)|} \] tends to infinity for \( t = \pm \frac{r}{c} \)
- amplification of high frequency energy due to temporal aliasing
- soft-knee limiting function with empirically chosen \( A_{\text{thr}} \)

\[ L(u) = \frac{2A_{\text{thr}}}{\pi} \arctan \left( \frac{\pi u}{2A_{\text{thr}}} \right) \]

- low-pass filtering \( \rightarrow \) reduce temporal aliasing artifacts
Dynamic Range Control
only for CHT

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$$L(u) = \frac{2A_{\text{thr}}}{\pi} \arctan \left( \frac{\pi u}{2A_{\text{thr}}} \right)$$

- low-pass filtering $\rightarrow$ reduce temporal aliasing artifacts
Frequency-Domain Driving Functions [Hahn et al., 2016]

\[ D_{2D}(x_0, \omega) = -2a(x_0) \sum_{\mu=-N}^{N} \tilde{S}_\mu(\omega)e^{i\mu\varphi_0} \]
\[ \times \left[ \langle \hat{\mathbf{e}}_r, n_0 \rangle \frac{\omega}{2c} \left\{ J_{\mu-1} \left( \frac{\omega}{c} \rho_0 \right) - J_{\mu+1} \left( \frac{\omega}{c} \rho_0 \right) \right\} + \langle \hat{\mathbf{e}}_\varphi, n_0 \rangle i\mu J_{\mu} \left( \frac{\omega}{c} \rho_0 \right) \right] \]

\[ D_{3D}(x_0, \omega) = -2a(x_0) \sum_{n=0}^{N} \sum_{m=-n}^{n} \tilde{S}^m_n(\omega) \]
\[ \left[ \langle \hat{\mathbf{e}}_r, n_0 \rangle \frac{1}{2n+1} \frac{\omega}{c} \left\{ nj_{n-1} \left( \frac{\omega}{c} r_0 \right) - (n+1)j_{n+1} \left( \frac{\omega}{c} r_0 \right) \right\} Y^m_n(\theta_0, \phi_0) \right. \]
\[ + \langle \hat{\mathbf{e}}_\theta, n_0 \rangle \frac{-1}{r_0 \sin^2 \theta_0} j_n \left( \frac{\omega}{c} r_0 \right) \left\{ (n+1)\cos \theta_0 Y^m_n(\theta_0, \phi_0) - \sqrt{\frac{2n+1}{2n+3}} ((n+1)^2 - m^2) Y^m_{n+1}(\theta_0, \phi_0) \right\} \]
\[ + \langle \hat{\mathbf{e}}_\varphi, n_0 \rangle \frac{im}{r_0 \sin \theta_0} j_n \left( \frac{\omega}{c} r_0 \right) Y^m_n(\theta_0, \phi_0) \right] \]

\[ D_{2.5D}(x_0, \omega) = \sqrt{\frac{2\pi ||x - x_0||}{i\frac{\omega}{c}}} \times D_{2D}(x_0, \omega) \]

\[ D_{2.5D}(x_0, \omega) = -2a(x_0) \sum_{m=-N}^{N} \frac{\tilde{S}^m_0(\omega)e^{im\varphi_0}}{4\pi i |m|-m Y^{-m}_{|m|} \left( \frac{\pi}{2} , 0 \right)} \]
\[ \times \left[ \langle \hat{\mathbf{e}}_r, n_0 \rangle \frac{\omega}{2c} \left\{ J_{m-1} \left( \frac{\omega}{c} \rho_0 \right) - J_{m+1} \left( \frac{\omega}{c} \rho_0 \right) \right\} + \langle \hat{\mathbf{e}}_\varphi, n_0 \rangle im J_m \left( \frac{\omega}{c} \rho_0 \right) \right] \]